An Extremal Problem for *H*-Linked Graphs

Alexandr Kostochka[1] **and Gexin Yu**[2]

[1]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS URBANA, ILLINOIS 61801 AND INSTITUTE OF MATHEMATICS NOVOSIBIRSK 630090, RUSSIA
E-mail: kostochk@math.uiuc.edu

[2]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS URBANA, ILLINOIS 61801
E-mail: gexinyu@uiuc.edu

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Abstract: We introduce the notion of *H*-linked graphs, where *H* is a fixed multigraph with vertices $w_1, \ldots, w_m$. A graph *G* is *H-linked* if for every choice of vertices $v_1, \ldots, v_m$ in *G*, there exists a subdivision of *H* in *G* such that v_i is the branch vertex representing w_i (for all *i*). This generalizes the notions of *k*-linked, *k*-connected, and *k*-ordered graphs. Given *k* and $n \geq 5k+6$, we determine the least integer *d* such that, for every loopless graph *H* with *k* edges and minimum degree at least two, every *n*-vertex graph with minimum degree at least *d* is *H*-linked. This value $D_1(k,n)$ appears to equal the least integer d' such that every *n*-vertex graph with minimum degree at least d' is *k*-connected. On the way to the proof, we extend a theorem by Kierstead et al. on the least integer d'' such that every *n*-vertex graph with minimum degree at least d'' is *k*-ordered.
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1. INTRODUCTION

Let H be a multigraph. An *H-subdivision* in a graph G is a pair (f, g) of mappings, where f maps $V(H)$ into $V(G)$ and g maps $E(H)$ into the set of paths in G such that:

(a) $f(u) \neq f(v)$ for all distinct $u, v \in V(H)$;
(b) for every $uv \in E(H)$, $g(uv)$ is an $f(u)$,$f(v)$-path in G, and distinct edges map into internally disjoint paths in G.

A graph G is *H-linked* if every injective mapping $f : V(H) \to V(G)$ can be extended to an H-subdivision in G. This is a natural generalization of k-linkage.[1]

Recall that a graph is *k-linked* if for every list of $2k$ vertices $\{s_1, \dots, s_k, t_1, \dots, t_k\}$, there exist internally disjoint paths $P_1, \dots, P_k$ such that each P_i is an s_i, t_i-path. From the definitions of k-linked and H-linked graphs, we immediately see that *a graph G is k-linked if and only if G is H-linked for every graph H with $|E(H)| = k$.*

It is known that to check that a graph on at least $2k$ vertices is k-linked, it is enough to check only the lists $\{s_1, \dots, s_k, t_1, \dots, t_k\}$, where all s_i and t_i are distinct. Thus, *a graph G on at least $2k$ vertices is k-linked if and only if G is M_k-linked, where M_k is the matching with k edges.*

Let B_k denote the (multi)graph with 2 vertices and k parallel edges. By Menger's Theorem, *a simple graph G on at least $k + 1$ vertices is k-connected if and only if G is B_k-linked.*

A graph is *k-ordered*, if for every ordered sequence of k vertices, there is a cycle that encounters the vertices of the sequence in the given order. Let C_k denote the cycle of length k. Clearly, *a simple graph G is k-ordered if and only if G is C_k-linked.*

After Chartrand introduced the notion of k-ordered graphs, several authors (see, e.g., [4, 5, 7, 10, 13]) studied sufficient degree conditions for a graph to be k-ordered. Recall that Dirac [2] found sufficient conditions for a simple graph G to be Hamiltonian in terms of the minimum degree, $\delta(G)$, and Ore [14] found similar conditions in terms of $\sigma_2(G)$, the minimum value of the sum $\deg(u) + \deg(v)$ over all pairs $\{u, v\}$ of non-adjacent vertices in G. Let $D_0(n, k)$ denote the minimum positive integer d such that every n-vertex simple graph with minimum degree at least d is k-ordered. Similarly, let $R_0(n, k)$ denote the minimum positive integer r such that every n-vertex simple graph G with $\sigma_2(G) \geq r$ is k-ordered. Improving on results in [4, 13], it was shown in [5] that $R_0(n, k) = n + \lceil (3k - 9)/2 \rceil$ for every $3 \leq k \leq n/2$. This implies that $D_0(n, k) \leq$

[1]After the paper was submitted, the authors learned that Ferrara, Gould, Tansey, and Whalen [6] also introduced and studied this notion.

$\lceil (2n+3k-9)/4 \rceil$ for every $3 \le k \le n/2$. Moreover, Kierstead, Sárközy, and Selkow [10] showed that $D_0(n,k) = \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ for $3 \le k \le (n+3)/11$. Observe that these bounds demonstrate the interesting phenomenon: $R_0(n,k) > 2D_0(n,k)$ for k small with respect to n. It is also known that $D_0(n,k) > \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ for $k > n/3$, but the value of $D_0(n,k)$ was not known for $(n+3)/11 < k < (2n)/5$. Kierstead et al. [10] asked about the value of $D_0(n,k)$ in this range for k.

The main result of our paper gives the minimum degree conditions for a graph to be H-linked if $\delta(H) \ge 2$.

Theorem 1. *Let H be a loopless graph with k edges and $\delta(H) \ge 2$. Every simple graph G of order $n \ge 5k+6$ with $\delta(G) \ge \lceil (n+k)/2 \rceil - 1$ is H-linked. If H is the cycle C_k with k edges, then every graph G of order $n \ge 5k+6$ with $\delta(G) \ge \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ is H-linked. The minimum degree conditions are sharp.*

This theorem extends the result of Kierstead et al. [10] in two directions: for a larger scope of k and for much more general H. In particular, Theorem 1 yields $D_0(n,k) = \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ for $k \le (n-6)/5$.

Observe that the restriction $\delta(G) \ge \lceil (n+k)/2 \rceil - 1$ is exactly the minimum degree condition that provides the k-connectivity of G. Thus, an evident degree condition for a graph to be k-connected provides that a graph is H-linked for many H. If one drops the condition $\delta(H) \ge 2$, then this degree restriction is not sufficient in general. In this case, one needs a higher minimum degree for many graphs H. Kawarabayashi and we [9] considered similar problem for k-linked graphs. Let $D(n,k)$ be the minimum positive integer d such that every n-vertex simple graph with minimum degree at least d is k-linked. Also, let $R(n,k)$ denote the minimum positive integer r such that every n-vertex simple graph G with $\sigma_2(G) \ge r$ is k-linked.

Theorem 2 ([9]). *If $k \ge 2$, then*

$$R(n,k) = \begin{cases} 2n-3 & n \le 3k-1; \\ \lfloor \frac{2(n+5k)}{3} \rfloor - 3 & 3k \le n \le 4k-2; \\ n+2k-3 & n \ge 4k-1, \end{cases} \tag{1}$$

and

$$D(n,k) = \left\lceil \frac{R(n,k)}{2} \right\rceil = \begin{cases} n-1 & n \le 3k-1; \\ \lfloor \frac{n+5k}{3} \rfloor - 1 & 3k \le n \le 4k-2; \\ \lceil \frac{n-3}{2} \rceil + k & n \ge 4k-1. \end{cases} \tag{2}$$

Note that $D(n,k) = \lceil R(n,k)/2 \rceil$ for all possible n and k, unlike the situation with $D_0(n,k)$ and $R_0(n,k)$. Egawa, Faudree, Györi, Ishigami, Schelp, and Wang [3] considered a closely related problem, but the answers differ, especially for $\sigma_2(G)$. The bounds of Theorem 2 and of Egawa et al. [3] are helpful in estimating $f(k)$—the minimum positive integer f such that every f-connected graph is k-linked. After a series of papers by Jung [8], Larman and Mani [11], Mader [12], and Robertson and Seymour [15], the first linear upper bound for f, namely, $f(k) \leq 22k$, was proved by Bollobás and Thomason [1]. Very recently, Thomas and Wollan [16] improved this bound to $f(k) \leq 16k$. Their proof is elegant. In [9], we show how to apply Theorem 2 in the Thomas–Wollan proof to improve their bound to $f(k) \leq 12k$. Using the idea in [9] among other new ideas, Thomas and Wollan improved the bound further to $f(k) \leq 10k$.

It is worth to mention that while the restriction on H to have the minimum degree at least 2 decreases the minimum degree in G providing that G is H-linked from about $0.5n + k$ to $0.5n + 0.5k$, further restrictions on the minimum degree of H do not affect the bound anymore.

Note also that formally the papers [4, 5, 7, 10, 13] discussed a stronger than being k-ordered notion of a *k-ordered Hamiltonian graph*, i.e., a graph in which for every ordered sequence of k vertices, there is a *Hamiltonian* cycle that encounters the vertices of the sequence in the given order. But in each of the papers, the main difficulty was to prove that a graph is k-ordered. It is not difficult to prove that every n-vertex k-ordered graph G with $\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ is k-ordered Hamiltonian (see, e.g., [10]). This fact, together with Theorem 1, yields that for $3 \leq k \leq (n-6)/5$, every n-vertex simple graph G with $\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ is k-ordered Hamiltonian.

We will use the following analog of the Hamiltonian property: Given two graphs H and G, we say that G is *fully H-linked* if every injective mapping $f : V(H) \to V(G)$ can be extended to an H-subdivision in G that contains all vertices of G. Clearly, a graph G is fully C_k-linked exactly when it is k-ordered Hamiltonian. The following lemma elaborates some ideas of a similar result in [3].

Lemma 3. *Let H be a loopless graph with k edges and $\delta(H) \geq 2$. If an n-vertex simple graph G is H-linked and $\sigma_2(G) \geq n + k - 2$, then G is fully H-linked.*

This lemma and Theorem 1 together immediately imply the following.

Theorem 4. *Let H be a loopless graph with k edges and $\delta(H) \geq 2$. Every simple graph G of order $n \geq 5k + 6$ with $\delta(G) \geq \lceil (n+k)/2 \rceil - 1$ is fully H-linked. Every graph G of order $n \geq 5k + 6$ with $\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ is fully C_k-linked.*

We will prove the upper bounds in Theorem 1 for general H and for $H = C_k$ in parallel, because the argument works in both cases. In the next section, we initialize the proof of the upper bounds for Theorem 1 by contradiction. We assume that there is no good H-linkage for some choice of branching vertices in G and consider an optimal in some sense linkage with the vertex set X. In Section 3, we estimate $|X|$. In Section 4, we show that some vertices outside of X have many common neighbors. Then in Section 5, we use these facts to finish the proof of the upper bounds in Theorem 1. In Section 6, we prove Lemma 3 and thus Theorem 4. Finally, in Section 7, we show examples confirming that the bounds of Theorem 1 are tight and discuss possible strengthenings.

2. THE SETUP FOR THE PROOF OF THEOREM 1

Let $f: V(H) \to V(G)$ be an injective mapping and $W = f(V(H))$. Let $E(H) = \{e_j = u_j^0 v_j^0 : 1 \leq j \leq k\}$. Let $u_j = f(u_j^0)$ and $v_j = f(v_j^0)$. Since $\delta(H) \geq 2$, we have $|W| = |V(H)| \leq k$.

Define a partial H-linkage $\mathcal{C} = \bigcup_{j=1}^{k} P_j$, where each P_j is either $\{u_j, v_j\}$ or a u_j, v_j-path, such that

(1) $|X| \leq |W| + 2\alpha + 4$, where X is the set of vertices of the partial H-linkage, and α is the number of non-empty paths;
(2) the paths P_j's are pairwise internally disjoint and internally disjoint from W.

The family $\mathcal{C}_0$ of all empty paths (that is, each $P_j = \{u_j, v_j\}$) satisfies the properties (1) and (2) above with $X = W$ and $\alpha = 0$. Therefore, $\mathcal{C}_0$ is a partial H-linkage.

If all the P_j are non-empty, then the partial H-linkage is an H-subdivsion in G.

A partial H-linkage is *optimal*, if as many as possible of the P_j-s are non-empty and subject to this, $\mathcal{C}$ has as few vertices as possible.

Suppose for a contradiction that $\mathcal{C}$ is an optimal partial H-linkage, but $\mathcal{C}$ is not an H-subdivision. Let, for definiteness, P_k be empty and set $x = u_k$ and $y = v_k$. Let X denote the set of vertices of $\mathcal{C}$. Let $A = N(x) - X, B = N(y) - X$. Let $R = V(G) - (X \cup A \cup B)$. Note that each of $|A|$ and $|B|$ is at least

$$\delta(G) - (|X| - 2) \geq \frac{n + k - 3}{2} - (|W| + 2(k-1) + 4 - 2)$$
$$\geq \frac{(5k+6) + k - 3}{2} - 3k = \frac{3}{2} > 1.$$

It follows that we may choose distinct $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

For $v \in V(G)$, let $d_j(v)$ denote the number of neighbors of v 'inside' P_j plus $\beta_j = 1/\deg_H(u_j^0)$ if $u_j \in N_G(v)$ and plus $\gamma_j = 1/\deg_H(v_j^0)$ if $v_j \in N_G(v)$. For example, if $P_j = u_j w_1 w_2 v_j$, $\deg_H(u_j^0) = 3$ and v is adjacent to u_j and w_2 in P_j, then $d_j(v) = 4/3$. We will use the fact that

$$\sum_{j=1}^{k} d_j(v) = |N_G(v) \cap X| \quad \text{for all } v \in V(G). \tag{3}$$

Let l_p be the number of P_j's of length p for $p \geq 1$, and l_0 be the number of empty paths. Then

$$|X| = |W| + \sum_{p \geq 1} (p-1) l_p = \sum_{j=1}^{k} (\beta_j + \gamma_j) + \sum_{p \geq 1} (p-1) l_p \tag{4}$$

and

$$k = \sum_{p \geq 0} l_p = \alpha + l_0. \quad (5)$$

In the next section, we will prove that $|X|$ is rather small and use this in Section 4 to prove that every two vertices in A (and every two vertices in B) have several common neighbors outside W. This will help us in Section 5 to construct more P_j-s than in $\mathcal{C}$, a contradiction to the choice of $\mathcal{C}$.

3. AN UPPER BOUND ON THE SIZE OF X

We will assume that every non-empty P_j is of the form $P_j = u_j, w_{1,j}, \ldots, w_{p_j-1,j}, v_j$. Sometimes, for simplicity, we will write p instead of p_j and w_i instead of $w_{i,j}$ if j is clear from the context. In the next three sections, for every $j = 1, \ldots, k$, we denote $\beta_j = 1/\deg_H(u_j^0)$, $\gamma_j = 1/\deg_H(v_j^0)$, $M_j = d_j(x) + d_j(y)$, and $L_j = d_j(a_1) + d_j(a_2) + d_j(b_1) + d_j(b_2)$.

Lemma 5. *For a $P_j = u_j, w_1, \ldots, w_{p-1}, v_j$, let $s_j = M_j + 0.5L_j$, $\beta = \beta_j$, and $\gamma = \gamma_j$. Define*

$$D_1(p, \beta, \gamma) = \begin{cases} p + 1 + 2\beta + 2\gamma & \text{for } p \leq 1, \\ p + 3 + 2\beta + 2\gamma & \text{for } p \geq 2. \end{cases}$$

Then (*a*) *$s_j \leq D_1(p, \beta, \gamma)$;*
(*b*) *$s_k \leq 2(\beta_k + \gamma_k)$. Furthermore, if $xy \notin E(G)$, then $s_k = \beta_k + \gamma_k$.*

Proof. Let $\lambda = \max\{\beta, \gamma\}$. Since $\delta(H) \geq 2$, we have $\lambda \leq 1/2$.

By the definition, $L_k = 2\beta_k + 2\gamma_k$. If $xy \in E(G)$, then $M_k = \beta_k + \gamma_k$; otherwise, $M_k = 0$. This proves (b).

Claim 3.1. *Let $Z = \{a_1, a_2, b_1, b_2\}$.*

- (*i*) *For each $z \in Z$, the distance in P_j between any two neighbors of z is at most two. In particular, each $z \in Z$ has at most 3 neighbors in P_j.*
- (*ii*) *If $p \geq 3$, then no $z \in Z$ is a common neighbor of u_j and v_j.*
- (*iii*) *If $p \geq 3$, then x and y have no interior neighbors at distance at most $p - 3$ in P_j.*
- (*iv*) *If $p \geq 3$, then x (respectively, y) has no interior neighbors at distance at most $p - 4$ in P_j from interior neighbors of b_1 and b_2 (respectively, of a_1 and a_2).*

Proof. If some $z \in Z$ is adjacent to w_i and w_{i+m} for some $m \geq 3$ (we treat u_j as w_0 and v_j as w_p), then we can replace P_j by a shorter u_j, v_j-path, a contradiction to the optimality of $\mathcal{C}$. This proves (i), and (ii) is a partial case of (i).

If x and y have interior neighbors at distance at most $p-3$ in P_j, then we can delete P_j from $\mathcal{C}$ and add a shorter x,y-path. This proves (iii). The same trick proves (iv). ■

In order to prove (a), we consider several cases (depending on p).

Case 1. $p=0$. Since $\mathcal{C}$ is optimal, each $z \in Z$ is adjacent to at most one of x and y. Therefore, $L_j \le 4\lambda \le 2$, and $s_j = M_j + 0.5L_j \le 2(\beta+\gamma)+1 = D_1(0,\beta,\gamma)$.

Case 2. $p=1$. Trivially,

$$s_j \le 2(\beta+\gamma)+0.5(4(\beta+\gamma)) \le 2(\beta+\gamma)+2 = D_1(1,\beta,\gamma).$$

Case 3. $p=2$. If each of x and y is adjacent to w_1 and some $z \in Z$ is adjacent to both u_j and v_j, then $\mathcal{C}$ is not optimal: we can replace P_j by the path u_j, z, v_j and add the path xw_1y. Otherwise, either $M_j \le 2(\beta+\gamma)+1$ and hence

$$s_j \le 2(\beta+\gamma)+1+0.5(4(\beta+\gamma+1)) \le 2(\beta+\gamma)+5 = D_1(2,\beta,\gamma),$$

or $L_j \le 4(\lambda+1)$ and hence

$$s_j \le 2(\beta+\gamma+1)+0.5(4(\lambda+1)) \le 2(\beta+\gamma)+2+3 = D_1(2,\beta,\gamma).$$

Case 4. $p=3$. By (iii), $M_j \le 2(\beta+\gamma)+2$. If $L_j \le 8$, then $s_j \le D_1(3,\beta,\gamma)$. Otherwise, because of the symmetry between A and B, we may assume that $d_j(a_1)+d_j(a_2) > 4$ and that $d_j(a_1) > 2$. Then by (ii), we may assume that a_1 is adjacent to w_1, w_2 and v_j and that a_2 is adjacent to w_1 and w_2 (and may be to one more vertex). If $yw_2 \in E(G)$, then we can replace P_j with u_j, w_1, a_1, v_j and add the path x, a_2, w_2, y, a contradiction to the optimality of $\mathcal{C}$. If neither of x and y is adjacent to w_2, then by (iii), $M_j \le 2(\beta+\gamma)+1$, by (ii), $L_j \le 4(2+\lambda) \le 10$, and therefore $s_j \le 2(\beta+\gamma)+6 = D_1(3,\beta,\gamma)$. If $xw_2 \in E(G)$ and some $b \in \{b_1, b_2\}$ is adjacent to w_2, then we can replace P_j with u_j, w_1, a_1, v_j and add the path x, w_2, b, y. Finally, if neither of b_1w_2 and b_2w_2 is in $E(G)$, then by (i) $d_j(b_1)+d_j(b_2) \le 2(1+\lambda) \le 3$, and hence by (ii) $L_j \le 5+3 = 8$.

Case 5. $p \ge 4$. If x has r interior neighbors and $r \ge 2$, then by (iii), $d_j(y) \le \beta+\gamma$ and by (iv), $d_j(b_i) \le \max\{0, 3-r\}+\lambda$. Thus in this case

$$s_j \le 2\beta+2\gamma+r+3+\max\{0,3-r\}+\lambda.$$

If $r \ge 3$, then $s_j \le 2\beta+2\gamma+p-1+3+\lambda \le p+3+2\beta+2\gamma = D_1(p,\beta,\gamma)$. If $r=2$, then $s_j \le 2\beta+2\gamma+r+4+\lambda \le 2\beta+2\gamma+p+2.5 \le D_1(p,\beta,\gamma)$, again.

Thus, we can assume that each of x and y has at most one interior neighbor in P_j. By (iv), $d_j(a_i)+d_j(y) \le \beta+\gamma+\lambda+3$ and $d_j(b_i)+d_j(x) \le \beta+\gamma+\lambda+3$

for $i = 1, 2$. Therefore, $s_j \leq 2\lambda + 6 + 2\beta + 2\gamma \leq 2\beta + 2\gamma + p + 2 + 1 = D_1(p, \beta, \gamma)$. This finishes the proof of (a). ∎

Lemma 6. *Let $Z = \{a_1, a_2, b_1, b_2\}$ and $V_0 = (A \cup B) - Z - N_G(Z)$. Then $|X| \leq |W| + 2\alpha + 1 - |R| - |V_0|$.*

Proof. Let $\Sigma' = \deg_G(x) + \deg_G(y) + (1/2)(\deg_G(a_1) + \deg_G(a_2) + \deg_G(b_1) + \deg_G(b_2))$. Observe that every vertex $w \notin X$ contributes to Σ' at most 2: If $w \in R$, then it is not adjacent to x and y, and if $w \in A$ (respectively, $w \in B$), then it is not adjacent to y, b_1, and b_2 (respectively, to x, a_1, and a_2). By the definition, every vertex in V_0 is not adjacent to any vertex in Z, and therefore contributes to Σ' at most 1. Furthermore, every $z \in Z$ contributes to Σ' at most 1.5, since it is not adjacent to itself. Therefore,

$$\Sigma' \leq 4 \cdot 1.5 + 2(|A \cup B| - 4) + 2|R| + \sum_{j=1}^{k} s_j - |V_0|. \tag{6}$$

By Lemma 5,

$$\begin{aligned} \sum_{j=1}^{k} s_j &\leq l_0 + 2l_1 + \sum_{p \geq 2}(p+3)l_p + 2\sum_{j=1}^{k}(\beta_j + \gamma_j) - 1 \\ &= l_0 + 2l_1 + \sum_{p \geq 2}(p+3)l_p + 2|W| - 1. \end{aligned} \tag{7}$$

Therefore,

$$\begin{aligned} \Sigma' &\leq 2(|A \cup B| + |R|) - |V_0| + 2(|W| + l_0 + \sum_{p \geq 1} p l_p) - 3 - l_0 \\ &\quad + \sum_{p \geq 2}(3-p)l_p = 2(n+k) - |V_0| - 3 + l_2 - l_0 - \sum_{p \geq 3}(p-3)l_p. \end{aligned} \tag{8}$$

If H is a cycle, then every β_j and γ_j is equal to 0.5, and the part (b) of Lemma 5 will deduct an additional 1 from (7). In this case, we will get $\sum_{j=1}^{k} s_j \leq l_0 + 2l_1 + \sum_{p \geq 2}(p-1)l_p + 2|W| - 2$ and will have -4 instead of -3 in (8).

Recall that for general H, we have $\Sigma' \geq 4\delta(G) \geq 2(n+k-2)$. Comparing with (8), we get $l_2 \geq l_0 + \sum_{p \geq 3}((p-3)l_p) - 1 + |V_0|$.

If H is a cycle, then we have only $\Sigma' \geq 4\delta(G) \geq 2(n+k-3)$, but because of -4 instead of -3 in (8), we have

$$l_2 \geq l_0 + \sum_{p \geq 3}((p-3)l_p) - 2 + |V_0|, \tag{9}$$

i.e., (9) holds in both cases.

Thus, by (4), (5), and (9),

$$|X| = \sum_{p\geq 1}(p-1)l_p + |W| = |W| + 2\sum_{p\geq 1} l_p - 2l_1 - l_2 + \sum_{p\geq 3}(p-3)l_p$$
$$\leq |W| + 2\alpha - 2l_1 - l_2 + l_2 + 2 - l_0 - |V_0| \leq |W| + 2\alpha + 1 - |V_0|. \quad (10)$$

Therefore if some $u \in R$ has a neighbor $a_0 \in A$ and a neighbor $b_0 \in B$, then we can add to $\mathcal{C}$ the path $P_k = x, a_0, u, b_0, y$. The new set of paths will be a better partial linkage, since we increase α by 1 and the new X would have size at most $|W| + 2\alpha + 1 + 3 \leq |W| + 2(\alpha + 1) + 2$. This is a contradiction to the optimality of $\mathcal{C}$. Thus, $N(a) \cap N(b) \cap R = \emptyset$ for each $a \in A$ and $b \in B$. This means that every $w \in R$ contributes to Σ' at most 1, and (6) becomes

$$\Sigma' \leq 6 + 2(|A \cup B| - 4) + |R| + \sum_{j=1}^{k} s_j - |V_0|.$$

Accordingly, (9) and (10) become $l_2 \geq l_0 + \sum_{p\geq 3}((p-3)l_p) - 2 + |R| + |V_0|$ and

$$|X| \leq |W| + 2\alpha + 1 - |R| - |V_0|. \quad (11)$$

■

Lemma 7. $|A| + |B| > 2k.$

Proof. By (11), $|A| + |B| = n - (|X| + |R|) \geq n - (|W| + 2\alpha + 1) \geq 5k + 6 - 3k + 1 > 2k.$ ■

4. COMMON NEIGHBORS OF VERTICES IN *A* OUTSIDE OF *X*

Lemma 8. *Each $v \in V(G)$ is adjacent to at least 3 vertices in $A \cup B - V_0$. In particular, either v has 2 neighbors in A that belong or are adjacent to the set $\{a_1, a_2\}$, or 2 neighbors in B that belong or are adjacent to the set $\{b_1, b_2\}$.*

Proof. By Lemma 6, $|X| \leq |W| + 2\alpha + 1 - |R| - |V_0|$ and hence $|X| + |R| + |V_0| \leq 3k - 1$. On the other hand,

$$\delta(G) \geq \left\lceil \frac{5k+6}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1 = 3k + 2.$$

Thus each vertex has at least $(3k + 2) - (3k - 1) = 3$ neighbors in $V(G) - X - R - V_0$. ■

Let A'' (respectively, B'') denote the set of vertices in X having at least 2 neighbors in A (respectively, in B) that belong or are adjacent to the set $\{a_1, a_2\}$ (respectively, $\{b_1, b_2\}$). The above lemma yields that

$$A'' \cup B'' = X. \tag{12}$$

We will need the following analog of Lemma 5.

Lemma 9. *For a $P_j = u_j, w_1, \dots, w_{p-1}, v_j$, let $M_j = d_j(x) + d_j(y)$ and $L_j = d_j(a_1) + d_j(a_2) + d_j(b_1) + d_j(b_2)$. Let $S_j = M_j + L_j$, $\beta = \beta_j = 1/\deg_H(u_j^0)$, and $\gamma = \gamma_j = 1/\deg_H(v_j^0)$. Define*

$$D_2(p, \beta, \gamma) = \begin{cases} 2p + 1 + 3\beta + 3\gamma & \text{for } p \leq 1, \\ 2p + 3 + 3\beta + 3\gamma & \text{for } p \geq 2. \end{cases}$$

Then (a) $S_j \leq D_2(p, \beta, \gamma)$;
(b) $S_k \leq 3(\beta_k + \gamma_k)$. Furthermore, if $xy \notin E(G)$, then $S_k = 2(\beta_k + \gamma_k)$.

Proof. The proof follows the lines of that for Lemma 5, in particular, the argument for (b) is simply the same. Thus, we present here only the proof of (a).

As in the proof of Lemma 5, let $\lambda = \max\{\beta, \gamma\}$ and consider several cases depending on p. We will use Claim 3.1 several times.

Case 1. $p = 0$. As in the proof of Lemma 5, each $z \in Z$ has at most one neighbor in $\{u_j, v_j\}$. If neither of u_j and v_j is adjacent to each $z \in Z$, then $L_j \leq \beta + \gamma + 2\lambda \leq \beta + \gamma + 1$, and therefore $S_j = M_j + L_j \leq 2(\beta + \gamma) + (\beta + \gamma + 1) = D_2(0, \beta, \gamma)$. Thus, we can assume that v_j is adjacent to each $z \in Z$. By Lemma 8, u_j has a neighbor $w \in A \cup B$ adjacent to some $z \in Z$. Then adding to $\mathcal{C}$ the path $P_j = u_j, w, z, v_j$ creates a better partial linkage, a contradiction.

Case 2. $p = 1$. Trivially, $S_j \leq 6(\beta + \gamma) \leq 3(\beta + \gamma) + 3 = D_2(1, \beta, \gamma)$.

Case 3. $p = 2$. If neither of x and y is adjacent to w_1, then $S_j \leq 2(\beta + \gamma) + 4(1 + \beta + \gamma) \leq D_2(2, \beta, \gamma)$. Suppose that $xw_1 \in E(G)$ and $yw_1 \notin E(G)$. If neither of a_1 and a_2 is adjacent to both u_j and v_j or $b_1w_1 \notin E(G)$, then

$$L_j \leq \max\{2(1 + \beta + \gamma) + 2(1 + \lambda), 4(1 + \beta + \gamma) - 1\} \leq 6 + (\beta + \gamma)$$

and $M_j + L_j \leq 7 + 3(\beta + \gamma)$. Otherwise, if $b_1w_1 \in E(G)$ and, say, a_1 is adjacent to both u_j and v_j, then we can replace P_j with the path u_j, a_1, v_j and add to $\mathcal{C}$ the path x, w_1, b_1, y, a contradiction to the choice of $\mathcal{C}$.

Finally, suppose that each of x and y is adjacent to w_1. If some $z \in Z$ is adjacent to both u_j and v_j, then $\mathcal{C}$ is not optimal: we can replace P_j by the path u_j, z, v_j and add the path xw_1y. Thus, we assume that no $z \in Z$ is adjacent to both u_j and v_j. As in Case 1, if neither of u_j and v_j is adjacent to each $z \in Z$, then $L_j \leq 4 + \beta + \gamma + 2\lambda \leq \beta + \gamma + 5$, and therefore $S_j = M_j + L_j \leq 7 + 3(\beta + \gamma) = D_2(2, \beta, \gamma)$. Otherwise, if, say, v_j is adjacent to each $z \in Z$, then by Lemma 8, u_j has a neighbor $w \in A \cup B$ adjacent to some $z \in Z$. In this case, we can replace P_j with the path $P_j = u_j, w, z, v_j$ and add to $\mathcal{C}$ the path x, w_1, y, a contradiction to the optimality of $\mathcal{C}$.

Case 4. $p = 3$. By (iii), $M_j \leq 2(\beta+\gamma)+2$. If $L_j \leq 7+\beta+\gamma$, then $S_j \leq D_2(3,\beta,\gamma)$. Suppose that

$$L_j > 7+\beta+\gamma. \tag{13}$$

Case 4.1. *There exists some $z \in Z$, say, a_1 adjacent to two non-adjacent vertices of P_j, say, to u_j and w_2 and such that $a_2w_1 \in E(G)$.* If $b_1w_1 \in E(G)$, then we can replace P_j with u_j, a_1, w_1, v_j and add to $\mathcal{C}$ the path x, a_2, w_1, b_1, y. By Lemma 6, the new family will be a partial linkage, a contradiction to the optimality of $\mathcal{C}$. Thus, neither of b_1 and b_2 is adjacent to w_1. Then by (ii), $L_j \leq 2(1+\lambda)+2(2+\lambda) = 6+4\lambda$. Moreover, if at least one $z \in Z$ is not adjacent to u_j, then $L_j \leq 6+(\beta+\gamma)+2\lambda$, a contradiction to (13). Suppose now that every $z \in Z$ is adjacent to u_j and (13) holds. For this, we need $\{w_1a_1, w_2a_2, w_2b_1, w_2b_2\} \subset E(G)$. Then by Lemma 8, v_j has a neighbor $w \in A \cup B$ adjacent to some $z \in Z$. By symmetry, we can assume that this z is b_1. In this case, we replace P_j by the path u_j, b_1, w, v_j and add to $\mathcal{C}$ the path x, a_1, w_2, b_2, y. This contradicts the choice of $\mathcal{C}$.

Case 4.2. *Case* 4.1 *does not hold.* By (ii), in this case, $d_j(a_1)+d_j(a_2) \leq 4$ and $d_j(b_1)+d_j(b_2) \leq 4$. Moreover, if Case 4.1 does not hold and, say, $d_j(a_1)+d_j(a_2) > 3+\beta+\gamma$, then either of a_1 and a_2 is adjacent to both w_1 and w_2. In view of (13), we derive that every $z \in Z$ is adjacent to both w_1 and w_2. Then by Lemma 8, v_j has a neighbor $w \in A \cup B$ either in Z or adjacent to some $z \in Z$. By symmetry, we can assume that this z is b_1. In this case, we replace P_j by the path $u_j, w_1, b_1, (w,)v_j$ and add to $\mathcal{C}$ the path x, a_1, w_2, b_2, y. This again contradicts the choice of $\mathcal{C}$.

Case 5. $p \geq 4$. If x has r interior neighbors and $r \geq 2$, then by (iii) $d_j(y) \leq \beta+\gamma$ and by (iv) $d_j(b_i) \leq \max\{0, 3-r\}+\lambda$. Thus in this case

$$S_j \leq 2\beta+2\gamma+r+6+2\max\{0,3-r\}+2\lambda.$$

If $r \geq 3$, then $S_j \leq 2\beta+2\gamma+p-1+6+2\lambda \leq p+6+2\beta+2\gamma \leq D_2(p,\beta,\gamma)$. If $r=2$, then $S_j \leq 2\beta+2\gamma+2+6+2+2\lambda \leq 2\beta+2\gamma+11 \leq D_2(p,\beta,\gamma)$, again.

Thus, we can assume that each of x and y has at most one interior neighbor in P_j. By (iv), $d_j(a_1)+d_j(y) \leq \beta+\gamma+\lambda+3$ and $d_j(b_1)+d_j(x) \leq \beta+\gamma+\lambda+3$. Using this and (i), we have $S_j \leq 2(\beta+\gamma+\lambda+3)+6 \leq 2(\beta+\gamma)+13$. This is at most $D_2(p,\beta,\gamma)$ for $p \geq 5$. Let $p=4$.

Case 5.1. *Some $z \in Z$, say, a_1, is adjacent to w_1 and w_3.* Then by (i) and (iv), $d_j(y)+d_j(a_1) \leq 3+\beta+\gamma$. Suppose that $a_2w_2 \in E(G)$. If neither of b_1 and b_2 is adjacent to w_2, then by (i), $d_j(b_1)+d_j(b_2) \leq 4$ and therefore $S_j \leq 3+(3+\beta+\gamma)+4+(1+\beta+\gamma) < D_2(4,\beta,\gamma)$. Thus, we may assume that $b_1w_2 \in E(G)$. Then we can replace P_j by the path u_j, w_1, a_1, w_3, v_j and add to $\mathcal{C}$ the path x, a_2, w_2, b_1, y, a contradiction to the optimality of $\mathcal{C}$.

Suppose now that $a_2w_2 \notin E(G)$. By (i), $d_j(a_2) \leq 2$, and therefore,

$$S_j \leq 3 + (\beta + \gamma) + 2 + (3 + \beta + \gamma + \lambda) + 3 \leq 11 + 2(\beta + \gamma) + \lambda \leq D_2(4, \beta, \gamma).$$

Case 5.2. *No $z \in Z$ is adjacent to both w_1 and w_3.* This yields, in particular, that $d_j(z) \leq 2 + \lambda$ for every $z \in Z$. Assume that $\lambda = \beta$. Recall that $d_j(a_1) + d_j(y) \leq \beta + \gamma + \lambda + 3$. Moreover, in our case, by (iv), if $d_j(a_1) + d_j(y) > \beta + \gamma + \gamma + 3$, then $u_ja_1 \in E(G)$ and $w_3y \in E(G)$. Since both x and y cannot be adjacent to w_3 by (iii), we have

$$(d_j(a_1) + d_j(y)) + (d_j(b_1) + d_j(x)) \leq 2(\beta + \gamma + 3) + \beta + \gamma$$

and therefore

$$S_j \leq 3(\beta + \gamma) + 6 + 2(2 + \lambda) \leq D_2(4, \beta, \gamma).$$

This finishes the proof of the lemma. ■

Lemma 10. *For every non-adjacent $s, t \in A$ (or B), $|N(s) \cap N(t) - X| \geq 3$.*

Proof. Suppose to the contrary that $a_1, a_2 \in A$, $a_1a_2 \notin E(G)$ and the cardinality of the set T of common neighbors of a_1 and a_2 outside of X is at most two. Consider arbitrary $b_1, b_2 \in B$ and let $Z = \{a_1, a_2, b_1, b_2\}$. Then the sum of degrees of vertices in Z in the subgraph of G induced by $Z \cup \{x, y\}$ is at most 6. Furthermore, for each $u \in A - Z - T$, $|N(u) \cap \{x, y\}| \leq 1$ and $|N(u) \cap Z| \leq 1$, and for each $u \in R$, $N(u) \cap \{x, y\} = \emptyset$ and $|N(u) \cap Z| \leq 2$. It follows that for $\Sigma'' = \deg_G(x) + \deg_G(y) + \deg_G(a_1) + \deg_G(a_2) + \deg_G(b_1) + \deg_G(b_2)$ we have

$$\Sigma'' \leq 6 + 2(|A| - 4) + 6 + 3(|B| - 2) + 2|R| + \sum_{j=1}^{k} S_j. \tag{14}$$

By Lemma 9,

$$\begin{aligned} \sum_{j=1}^{k} S_j &\leq l_0 + 3l_1 + \sum_{p \geq 2}(2p + 3)l_p + 3\sum_{j=1}^{k}(\beta_j + \gamma_j) - 1 \\ &\leq l_0 + 3l_1 + \sum_{p \geq 2}(2p + 3)l_p + 3|W| - 1. \end{aligned} \tag{15}$$

Therefore,

$$\begin{aligned} \Sigma'' \leq 3(|A| + |B| + |R| + |W| + l_0 + \sum_{p \geq 1} pl_p) - 3 - 2l_0 + l_2 - \sum_{p \geq 3}(p - 3)l_p \\ -|A| - |R| \leq 3(n + k) + l_2 - 3 - 2l_0 - |A| - |R| - \sum_{p \geq 3}(p - 3)l_p. \end{aligned} \tag{16}$$

Recall that $\Sigma'' \geq 6\delta(G) \geq 3(n+k-3)$. Thus by (4) and (5),

$$\begin{aligned} |X| &= |W| + \sum_{p\geq 1}(p-1)l_p = |W| + 2\alpha + \sum_{p\geq 3}(p-3)l_p - 2l_1 - l_2 \\ &\leq |W| + 2\alpha + l_2 - (|A| + |R| + 2l_0) - 2l_1 - l_2 + 6 \\ &\leq k + 2(k-1) - |A| - |R| - 2l_0 + 6, \end{aligned} \tag{17}$$

that is,

$$|A| + |R| + |X| \leq 3k + 2.$$

Then for $a_1 \in A$, we have $\deg_G(a_1) \leq |A| + |R| + |X| - 2 \leq 3k$, which contradicts the minimum degree condition. ■

5. PROOF OF THEOREM 1

Lemma 11. *Let X be optimal, $j \in [k]$, and either $\{u_j, v_j\} \subset A''$ or $\{u_j, v_j\} \subset B''$. Then for each $a \in A$ and $b \in B$,*

$$(N(a) \cap N(b) \cap P_j)\backslash\{u_j, v_j\} = \emptyset.$$

Proof. Assume to the contrary that $r \in N(a) \cap N(b) \cap P_j\backslash\{u_j, v_j\}$. Let $P'_k = (x, a, r, b, y)$. Without loss of generality, assume that $\{u_j, v_j\} \subset A''$. Then there exist $s \in N(u_j) \cap A\backslash\{a\}$ and $t \in N(v_j) \cap A\backslash\{a\}$. If $s = t$ or s is adjacent to t, then let $P'_j = (u_j, s, t, v_j)$.

If s and t are non-adjacent, then by Lemma 10, we have $|(N(s) \cap N(t))\backslash X| \geq 3$, and therefore there exists $q \in N(s) \cap N(t)\backslash(X \cup \{a, b\})$. In this case, let $P'_j = (u_j, s, q, t, v_j)$. In both cases, P'_j is a path disjoint from P'_k. Thus in both cases, we increase α by one and, by (11), maintain $|X| \leq |W| + 2\alpha + 4$. This is a contradiction. ■

Similarly to $d_j(v)$, let $d_j(u, v)$ denote the number of common neighbors of u and v 'inside' P_j plus $\beta_j \cdot |N(v) \cap \{u_j\}|$ plus $\gamma_j \cdot |N(v) \cap \{v_j\}|$. Let X be optimal, $a \in A$, $b \in B$. Since $x, y \notin N(a) \cap N(b)$, we have $N(a) \cap N(b) \cap (V(G) - X + x + y) = \emptyset$. For general H and for even k when H is a cycle, $|N(a) \cap N(b)| \geq 2(n+k-2)/2 - (n-2) = k$. It follows that there exists some $j = j(a, b) \in [k-1]$ such that $d_j(a, b) > 1$. If H is a cycle and k is odd, then we have only $|N(a) \cap N(b)| \geq k-1$, but there is some $h \in [k-1]$ such that $\{u_h, v_h\} \subset A''$ or $\{u_h, v_h\} \subset B''$. In this case, by Lemma 11, $N(a) \cap N(b) \cap P_h\backslash\{u_h, v_h\} = \emptyset$ and among the remaining $k-2$ indices, there exists some $j = j(a, b) \in [k]$ such that $d_j(a, b) > 1$. In the rest of this section, the choice of $j(a, b)(a \in A,\ b \in B)$ is fixed.

Lemma 12. *Let X be optimal, $j \in [k]$. Then there is at most one $a \in A$, such that there is more than one $b \in B$ with $j = j(a, b)$.*

Proof. Assume to the contrary that there are $a_1, a_2 \in A$ and $b_1, b_2, b_3, b_4 \in B$ such that $j(a_1, b_1) = j(a_1, b_2) = j(a_2, b_3) = j(a_2, b_4) = j$, where $a_1 \neq a_2$, $b_1 \neq b_2$, $b_3 \neq b_4$. Note that by the definition of $j(a, b)$, each of a_i, $i = 1, 2$ has a common neighbor with each of b_{2i-1} and b_{2i} among interior vertices of P_j. We may assume that $u_j \in A''\backslash B''$, $v_j \in B''\backslash A''$.

Case 1. Some of a_1, a_2 has at least two common neighbors with some b_i among interior vertices of P_j. Then there exist distinct $s_1, s_2 \in P_j\backslash\{u_j, v_j\}$ such that (a_1, s_1, b_1) and (a_2, s_2, b_3) are paths. Assume that the order in P_j is u_j, s_1, s_2, v_j.

Assume that $a'(\neq a_1)$ is a neighbor of u_j. If $a' = a_2$ or is adjacent to a_2, we have two disjoint paths x, a_1, s_1, b_1, y and $u_j, a', a_2, s_2, P_j, v_j$. Deleting P_j from $\mathcal{C}$ and adding these two paths will increase α and by (11) will maintain $|X| \leq |W| + 2\alpha + 4$. Otherwise, by Lemma 10, a' and a_2 have a common neighbor $a \in A' - \{a_1\}$. Then we have two disjoint paths x, a_1, s_1, b_1, y and $u_j, a', a, a_2, s_2, P_j, v_j$. As above, replacing P_j with these paths increases α and by (11) maintains $|X| \leq |W| + 2\alpha + 4$. This is a contradiction.

Case 2. Each of $N(a_i) \cap N(b_l)$, $i = 1, 2$, $l = 2i - 1, 2i$, contains exactly one internal vertex of P_j and some of u_j, v_j (may be both). Since $v_j \in B''\backslash A''$, we may assume that $u_j \in N(a_1) \cap N(b_1) \cap N(b_2)$. But this contradicts the fact that $u_j \in A''\backslash B''$. ∎

By Lemma 7, $|A| + |B| > 2k$. We may assume that $|A| \leq |B|$. Thus $|B| \geq k$. If $|A| \geq k$, then since $|B| \geq k$, for each $a \in A$ there is some $j(a)$ and distinct $b_1(a)$ and $b_2(a)$ such that $j(a) = j(a, b_1(a)) = j(a, b_2(a))$. Furthermore, since $|A| \geq k$, for some distinct $a_1, a_2 \in A$, the indices $j(a_1)$ and $j(a_2)$ are the same. This contradicts Lemma 12.

Thus we may assume that $|A| < k$. Since $|B| \geq k$, for each $a \in A$ there exists some $j(a)$ and distinct $b_1(a)$ and $b_2(a)$ such that $j(a) = j(a, b_1(a)) = j(a, b_2(a))$. Furthermore, by Lemma 12, the indices $j(a)$ are distinct for distinct $a \in A$. Let $J = \{j(a) \mid a \in A\}$. Note that $|J| = |A|$.

Lemma 13. *Suppose that $j \in J$ and $P_j = u_j, w_1, \ldots, w_{p-1}, v_j$. Then x has at most $p - 2$ interior neighbors in P_j.*

Proof. For every $j \in J$, by the definition of J, there exists $a \in A$ and distinct $b_1, b_2 \in B$ such that $d_j(a, b_1), d_j(a, b_2) > 1$. Since $\beta_j + \gamma_j \leq 1$, this implies that $p \geq 2$. Assume that $u_j \in A''$ and $v_j \in B''$. Let $a', a'' \in A$ be two neighbors of u_j and $b', b'' \in B$ be two neighbors of v_j.

Suppose that the lemma does not hold and $xw_i \in E(G)$, $1 \leq i \leq p - 1$. Assume that v_j is a common neighbor of a, b_1 and b_2. Let w be a common neighbor of a and b_1 "inside" P_j. By Lemma 10, there is a common neighbor, say

$a_1 \in V(G) - X$, of a and a'. Thus u_j, a', a_1, a, v_j and x, w, b_1, y are two disjoint paths. Replacing P_j in $\mathcal{C}$ by these paths increases α and, by (11), maintains $|X| \leq |W| + 2\alpha + 4$.

Therefore, we may assume that v_j is not a common neighbor of a, b_1 and b_2. Then there are distinct w', w'' inside P_j such that $aw', b_1w', aw'', b_2w'' \in E(G)$. Now by Lemma 10, there is a common neighbor, say $a_1 \in V(G) - X$, of a and a'. Then $u_j, a', a_1, a, w'', P_j, v_j$ and x, w', b_1, y are two disjoint paths. As above, replacing P_j with these two paths increases α and by (11), maintains $|X| \leq |W| + 2\alpha + 4$. This contradicts the optimality of $\mathcal{C}$. ■

End of the proof. By Lemma 13, x is not adjacent to at least $|J|$ vertices in $X - W$. Because it also is not adjacent to itself, we have $|N(x) \cap X| \leq |X| - |J| - 1 \leq (3k+2) - |J| - 1$. Since $|J| = |A| = |N(x) - X|$, we get

$$\frac{n+k-3}{2} \leq \deg(x) \leq 3k+1.$$

This is impossible if $n \geq 5k+6$. ■

6. FULLY *H*-LINKED GRAPHS

The proof of Lemma 3 uses ideas of proofs for similar statements in [3, 5, 7, 10], but needs some specific details.

Let H be a loopless graph with edges $e_1, \ldots, e_k$ and $\delta(H) \geq 2$. Then $k \geq 2$. Let f be any injection $V(H) \to V(G)$. Since G is H-linked, f can be extended to an H-subdivision in G. Among such subdivisions, choose one of the maximum order. Suppose that for every $i \in \{1, \ldots, k\}$, the edge e_i is mapped to a u_i, v_i-path P_i, and all paths $P_1, P_2, \ldots, P_k$ are internally vertex-disjoint.

Let $W := \cup_{i \leq k} V(P_i)$. Suppose that $W \neq V(G)$, and let G_1 be a component of $G - W$. Let $Z = N_W(V(G_1))$.

Claim 6.1. *There exists i such that $|Z \cap V(P_i)| \geq 2$.*

Proof. If $|Z| \geq k+1$, we are done by the pigeonhole principle. Let $U = W - Z$ and assume that $|Z \cap V(P_i)| \leq 1$ for every i. Then $|Z| \leq k$ and $U \neq \emptyset$.

Let $v \in V(G_1)$ and $w \in U$. Then $\deg(v) + \deg(w) \leq n + |Z| - 2$. Thus the degree condition yields $n + k - 2 \leq n - 2 + |Z|$. It follows that

(a) $|Z| = k$;
(b) every vertex of G_1 is adjacent to every vertex in $Z \cup V(G_1)$;
(c) every vertex in U is adjacent to every vertex in W.

Since $|Z| = k$, every P_i contains exactly one vertex in Z and no end vertex of any P_i is in Z, i.e., all end vertices of all P_i are in U. But then we construct a full linkage as follows.

Let $Z = \{z_1, \dots, z_k\}$. By (c), we can take P'_1 as the edge u_1v_1 and for $i = 2, 3, \dots, k-1$, denote $P'_i = u_i z_{i-1} v_i$. Finally, the path P'_k starts at u_k, then passes through all vertices of $U - \{u_i, v_i : i = 1, \dots, k\}$ to z_{k-1} (it can be done by (c)), then passes through all vertices of G_1 to z_k (it can be done by (b)), and finishes at v_k. This proves the claim. ∎

Among all P_i-s with at least two neighbors of G_1, choose a path P_j where the distance along the path between some two neighbors of G_1 is the smallest. We may assume that $j = 1$, $P_1 = s_0, s_1 \dots, s_r$ (where $s_0 = u_1$ and $s_r = v_1$) and the closest on P_1 neighbors of G_1 are s_p and s_q with $p < q$.

Call a path in G an *H-path* if it is a subpath of some path F that is the union of some P_i-s. Respectively, call a cycle an *H-cycle* if it is the union of some P_i-s.

Claim 6.2. *If F is an H-path or an H-cycle, then no two consecutive vertices on F are neighbors of G_1.*

Proof. If vertices u and w are consecutive on F, then they are consecutive on some P_j. If both u and w have neighbors in G_1, then we can enlarge this P_j replacing the edge uw in it by a u,w-path with internal vertices in $V(G_1)$. This contradicts the choice of P_i-s. ∎

By Claim 6.2, $q - p \geq 2$. Let $P = (s_1, \dots, s_p)$, $Q = (s_q, \dots, s_r)$, and $W' = W - \{s_i : p < i < q\}$. Take $v \in N(s_p) \cap V(G_1), v' \in N(s_q) \cap V(G_1)$. By the choice of p and q, $vs_{p+1} \notin E(G)$. Therefore,

$$\deg(v) + \deg(s_{p+1}) \geq n + k - 2. \tag{18}$$

Again, by the choice of p and q, $N(v) - W' \subseteq V(G_1) - \{v\}$ and $N(s_{p+1}) - W' \subseteq V(G) - W' - V(G_1) - \{s_{p+1}\}$. Thus

$$\begin{aligned}\deg_{V(G)-W'}(v) + \deg_{V(G)-W'}(s_{p+1}) &\leq |V(G_1)| - 1 \\ &\quad + n - |W'| - |V(G_1)| - 1 = n - |W'| - 2.\end{aligned}$$

By (17), $\deg_{W'}(v) + \deg_{W'}(s_{p+1}) \geq (n + k - 2) - (n - |W'| - 2) = |W'| + k$.

Similarly, $\deg_{W'}(v') + \deg_{W'}(s_{q-1}) \geq |W'| + k$. Thus,

$$\deg_{W'}(v) + \deg_{W'}(v') + \deg_{W'}(s_{p+1}) + \deg_{W'}(s_{q-1}) \geq 2|W'| + 2k. \tag{19}$$

In order to estimate $\deg_{W'}(s_{p+1}) + \deg_{W'}(s_{q-1})$ from above, we need the following analog of Claim 6.2.

Claim 6.3. *If F is an H-path or an H-cycle, then there are no two consecutive vertices u and w on $F \cap W'$ such that $us_{p+1} \in E(G)$ and $ws_{q-1} \in E(G)$.*

Proof. If vertices u and w are consecutive on $F \cap W'$, then they are consecutive on some P_i. If $us_{p+1} \in E(G)$ and $ws_{q-1} \in E(G)$, then we can modify

P_i by replacing the edge uw in it with the path $u, s_{p+1}, s_{p+2}, \ldots, s_{q-1}w$, and modify P_1 by deleting $s_{p+1}, s_{p+2}, \ldots, s_{q-1}$ and adding any s_p, s_q-path with internal vertices in $V(G_1)$. The modified set of paths would have more vertices than the original, a contradiction. ■

In view of (18), the following observation is important.

Claim 6.4. *If W' is the disjoint union of several H-cycles and at most $k-1$ H-paths, then* $\deg_{W'}(s_{p+1}) + \deg_{W'}(s_{q-1}) < |W'| + k$ *and* $\deg_{W'}(v) + \deg_{W'}(v') < |W'| + k$.

Proof. Suppose that $R = (x_1, \ldots, x_r)$ is an H-path in W'. By Claim 6.3, if vertices $x_{i_1}, \ldots, x_{i_h}$ are adjacent to s_{p+1}, then vertices $x_{i_1+1}, \ldots, x_{i_h+1}$ are not adjacent to s_{q-1}. It follows that $\deg_R(s_{p+1}) + \deg_R(s_{q-1}) \leq r + 1$ (the $+1$ arises because it might happen that $i_h = r$).

Similarly, if $R = (x_1, \ldots, x_r)$ is an H-cycle in W', then $\deg_R(s_{p+1}) + \deg_R(s_{q-1}) \leq r$ (no $+1$ arises in this case). Thus, if W' is the disjoint union of a H-cycles and b H-paths, then $\deg_{W'}(s_{p+1}) + \deg_{W'}(s_{q-1}) \leq |W'| + b$. This proves the first statement of the claim. The second statement follows exactly the same way with Claim 6.2 in place of Claim 6.3. ■

Suppose now that W is the disjoint union of a H-cycles and b H-paths. Recall that $W' = W - \{s_i : p < i < q\}$. If $\{s_i : p < i < q\}$ is a part of an H-cycle, then W' is the disjoint union of $a-1$ H-cycles and at most $b+1$ H-paths. Similarly, if $\{s_i : p < i < q\}$ is a part of an H-path, then W' is the disjoint union of a H-cycles and at most $b+1$ H-paths. This, together with (18) and Claim 6.4, yields that the lemma will be proved if we show that

$$W \text{ is the disjoint union of some } H\text{-cycles and at most } k-2H\text{-paths.} \tag{20}$$

Since $\delta(H) \geq 2$, H has a cycle, say, C. Suppose that the edges of C are $e_1, \ldots, e_l$, where $l \geq 2$. Let $Q_0 = \bigcup_{i=1}^{l} V(P_i)$ and, for $i = l+1, \ldots, k$, let $Q_{i-l} = V(P_i) - \bigcup_{j=0}^{i-1} Q_j$. By construction, Q_0 spans an H-cycle. Since all P_i-s are internally disjoint, each Q_i, $i = 1, \ldots, k-l$ spans a subpath of P_{i+l}, that is, an H-path. This proves (19), and thus the lemma.

7. EXAMPLES AND CONCLUSION

Let G be the n-vertex graph with $V(G) = V_0 \cup V_1 \cup V_2$ such that $G[V_1] = K_{\lceil (n-k+1)/2 \rceil}$, $G[V_2] = K_{\lfloor (n-k+1)/2 \rfloor}$, and all the vertices in V_0 (with $|V_0| = k-1$) are all-adjacent in G. Clearly, $\delta(G) = \lfloor (n+k-1)/2 \rfloor - 1$.

Let H be any bipartite graph with k edges and let X and Y be the partite sets in H. We claim that G does not contain a subdivision of H such that X is mapped into V_1 and Y is mapped into V_2. This is because every edge of H should be mapped into a V_1, V_2-path and thus should contain a vertex in V_0, but

$|V_0| = k - 1$. The same example shows the sharpness of the upper bound for $H = C_{k+1}$ when $k + 1$ is odd, because C_{k+1} has a bipartite subgraph with k edges.

The reader can find in [5, 10] examples showing that the minimum degree $(n + k)/(2) - 1$ of an n-vertex graph does not provide that this graph is k-ordered if $n < 3k - 6$. The proof of Theorem 1 can be elaborated so that the restriction $n \geq 5k + 6$ relaxes to $n \geq 5k - 1$, but we do not know exact values of $D_0(k, n)$ for $5k/2 < n < 5k - 2$.

While the minimum degree condition for a graph to be k-ordered is weaker than that for a graph to be k-linked, we do not know whether the same holds for connectivity conditions. It would be also interesting to find an analog of Theorem 1 for $\sigma_2(G)$ in place of $\delta(G)$. Recall that if $\sigma_2(G) \geq n + (3k - 9)/2$ and $k \geq 4$, then G is k-ordered. It would be interesting to derive an exact bound of this type for H-linked graphs G on n vertices if H is an arbitrary graph with k edges and minimum degree at least two.

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